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Residues and Poles of Complex Functions

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under the supervision of

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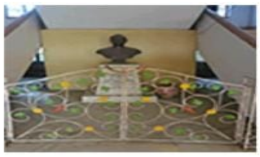

Certificate

This is to certify that the project entitled "Residues and Poles of Complex Functions ." which is being submitted by (Roll no. : 21031240012 Reg. No.;- 202101019444.) in partial fulfillment of the award of the B.Sc. degree in Mathematics is a record of bonafide project work carried out by his/her in the Department of Mathematics of Dr. Bhupendra Nath Dutta Smriti Mahavidyalaya under my supervision and guidance. The present project work has already reached the standard fulfilling the requirements of the regulations relating to the degree. The material of the project has not been submitted elsewhere for the award of any degree or diploma.

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1. INTRODUCTION

The Cauchy- Goursat theorem states that if a function is analytic at all points interior to and on a simple closed contour C, then the value of the integral of the function around that contour is zero. If, however, the function fails to be analytic at a finite number of points interior to C what will be the value of the integral of the function around that contour. We study here different types of complex functions which are not analytic, we study the integrals of that functions. We develop here the theory of residues; and we shall illustrate their use in certain areas of applied mathematics.

2. SOME PRELIMINARY DEFINITIONS

Definition :1 **ANALYTIC FUNCTIONS**

A function f is said to be analytic at a point z_0 if the function is differentiable at some neighborhood of z_0 i.e, there exists a neighborhood $|z-z_0|<\delta$ at all points of which f'(z) exists.

Definition :2 **TAYLOR SERIES**

Let f(z) be analytic inside and on a simple closed curve C. let a and a+h be two points inside C. then

$f(a+h) = f(a)+ hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} f^{(n)}(a) + ...$

Or writing z=a+h, h=z-a,

$f(z) =f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}$ (z-a)2 +.. . . . + $\frac{f^{(n)}(a)}{n!} (z-a)^n + ...$

this is called Taylor's series or expansion for f(a+h) or f(z).

EXAMPLE:1

Taylor Series expansion of $f(z) = \frac{1}{1-z}$ is

$$\sum_{n=0}^{\infty} -$$

Definition :3 **SINGULAR POINT**

A point at which f(z) fails to be analytic is called a Singular point of singularity of f(z).

Definition :4 **ISOLATED SINGULAR POINTS**

The point $z=z_0$ is called an isolated singular point of f(z) if we can find $\delta<0$ such that the circle $|z-z_0|=\delta$ encloses no singular point other than z_0.

EXAMPLE : 1

The function$(z+1)/z^3(z^2+1)$

has the three isolated points z=0 and z=±*i*

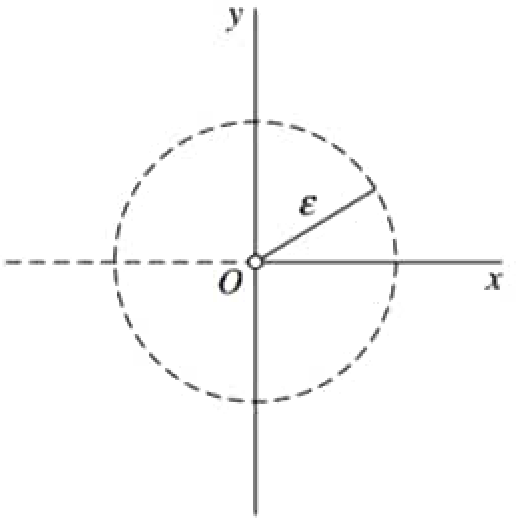
EXAMPLE: 2

The origin is a singular point of the principal branch

$\log z = \ln r + i\Theta \quad (r>0, -\pi< \Theta < \pi)$

of the logarithmic function . it is not however an isolated singular point since every deleted ε neighborhood of it contains points on the negative real axis and the branch is not even defined there. Similar remarks can be made regarding any branch

$\log z = \ln r + I\Theta \qquad (r>0, \alpha < \Theta < \alpha +2\pi)$ of the logarithmic function.



EXAMPLE: 3

The function $\dfrac{1}{\sin(\frac{\pi}{z})}$

has the singular points z=0 and z=1/n (n= ±1, ±2,....) , all lying on the segment of the real axis from z= -1 to z= 1. Each singular point z=0 is not isolated because every deleted ε neighborhood of then origin contains other singular points of the function. More precisely, when a positive number ε is specified and m is any positive integer such that m> 1/ ε, the fact that 0<1/m < ε means that the point z=1/m lies in the deleted ε neighborhood 0<|z|< ε.

Definition:5 **LAURENT SERIES**

The Laurent Series of a complex function f(z) is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions.

Laurent Series expansion is

$f(z)=a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots$

Definition :6 **MACLAURIN SERIES**

A Maclaurin Series is a power series that allows one to calculate an approximation of a function f(x) for input values close to zero.

EXAMPLE :1

Let us show that

$$\int_c \exp(1/z^2)\, dz = 0$$

when C is the same oriented circle |z|=1 . since 1/z2 is analytic everywhere except at the origin. The same is true of the integrand. The isolated singular point z=0 is interior to C. with the aid of the Maclaurin series representation

$$e^z = 1 + z/1! + z^2/2! + z^3/3! + \ldots \quad (|z|<\infty)$$

and the Laurent series expansion

$$\exp(1/z^2) = 1 + 1/1!.1/z^2 + 1/2!.1/z^4 + 1/3!.1/z^6 + \ldots \quad (0<|z|<\infty)$$

the residue of the integrand at its isolated singular point z=0 is , therefore, zero ($b_1=0$) and the value is established.

EXAMPLE :3

A residue can also be used to evaluate the integral

$$\int_C dz/z(z-2)^4$$

where C is the positively oriented circle |z-2|=1. Since the integrand is analytic everywhere in the finite plane except at the points z=0 and z=2, it has a Laurent Series representation that is valid in the punctured disk 0<|z-2|<2.

Now using Maclaurin series expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|<1)$$

and using it to write

$$1/z(z-2)^4 = 1/(z-2)^4 \cdot \frac{1}{2+(z-2)}$$

$$= 1/2(z-2)^4 \cdot \frac{1}{1-\left(-\frac{z-2}{2}\right)}$$

$$= \sum_{n=0}^{\infty} 1.\ (-1)^n/2n+1\ .\ (z-2)^{n-4} \qquad (0<|z-2|<2)$$

In this Laurent series, the coefficient of 1/(z-2) is the desired residue , namely -1/16. Consequently,

$$\int_C \frac{dz}{z(z-2)^4} = 2\pi i\left(-\frac{1}{16}\right) = -\frac{\pi i}{8}$$

3. RESIDUES

When z_0 is an isolated singular point of a function f, there is a positive number R_2 such that f is analytic at each point z for which $0<|z-z_0|< R_2$, consequently, f(z) has a Laurent Series representation.

1. $f(z) = \sum_{n=}^{\infty} a_n(z-z_0)n+\frac{b_1}{(z-z_0)}+\frac{b_2}{(z-z_0)^2}+.....+\frac{b_n}{(z-z0)^n}+......$ $(0<|z-z_0|<R_2)$

   where the coefficients a_n and b_n have certain integral representations.

   $b_n=\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$ (n=1,2….)

   where C is any positively oriented simple closed contour around z0 that lies in the punctured disk $0<|z-z_0|<R_2$. When n=1, this expression for b_n becomes

2. $\int_C f(z)dz = 2\pi i b_1$

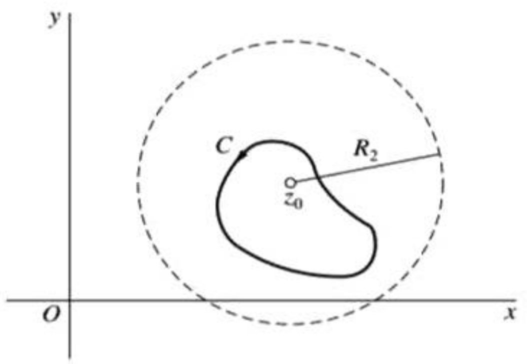
   the complex number b_1 ,which is the coefficient of $1/(z-z_0)$ is called the ***Residue*** of f at the isolated singular point z_0 and we shall ofter write

   $b_1=\text{Res}_{z=z0}\ f(z)$

   then equation 2. becomes

3. $\int_C f(z)= 2\pi i\ \text{Res}_{z=z_0} f(z)$

sometimes we simply use B to denote the residue when the function f and the point z_0 are clearly indicated.

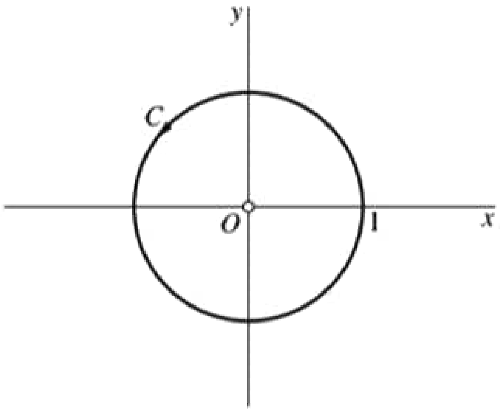


EXAMPLE : 1

Consider the integral

$$\int_C z^2 \sin(1/z)\ dz$$

where C is the positively oriented nit circle $|z|=1$. Since the integrand is analytic everywhere is the finite plane except at $z=0$, it has a Laurent Series representation that is valid when $0<|z|<\infty$. Thus the value of the integral is $2\pi i$ times the residue of its integrand at $z=0$.



To determine that residue , the Maclaurin series representation

$$\sin z = z - z^3/3! + z^5/5! + z^7/7! + \ldots (|z|<\infty)$$

and use it to write

$$z^2 \sin(1/z) = z - 1/3!.1/z + 1/5!.1/z^3 - 1/7!.\ 1/z^5 + \ldots\ (0<|z|<\infty)$$

the coefficient of $1/z$ here is the desired residue. Consequently,

$$\int_C z^2 \sin(1/z)\, dz = 2\pi i\,(-1/3!) = -\frac{\pi i}{3}$$

EXAMPLE :2

Let us show that

$$\int_C \exp\left(\frac{1}{z^2}\right) dz = 0$$

When C is the same oriented circle $|z|=1$. since a/z2 is analytic everywhere except at the origin, the same is true of the integrand. The isolated singular point $z=0$ is interior to C. with the aid of the Laurent series representation

$$\exp\left(\frac{1}{z^2}\right) = 1 + \frac{1}{1!}\cdot\frac{1}{z^2} + \frac{1}{2!}\cdot\frac{1}{z^4} + \frac{1}{3!}\cdot\frac{1}{z^6} + \ldots\ (|z|<\infty)$$

the residue of the integrand at its isolated singular point $z=0$ is therefore zero ($b_1=0$), and the value of integral is established.

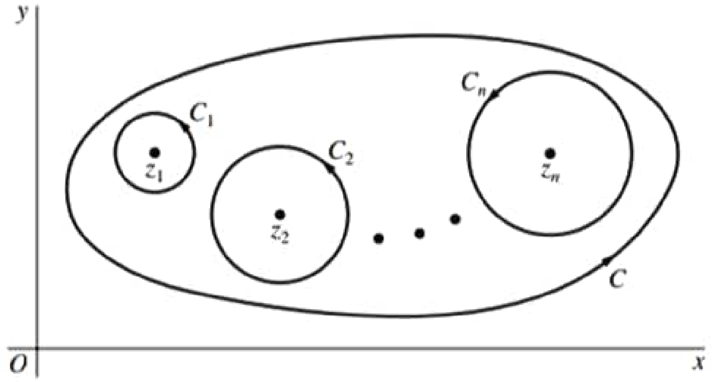
4. CAUCHY'S RESIDUE THEOREM

If, except for a *finite* number of singular points, a function *f* is *analytic* inside a simple closed contour *C*, those singular points must be isolated. The following theorem, which is known as `Cauchy's Residue Theorem`, is a precise statement of the fact that if f is also analytic on *C* and if *C* is positively oriented, then the value of the integral of f around C is $2\pi i$ times the sum of the residues of f at the singular points inside *C*.

THEOREM : 1

Let *C* be a simple closed contour, described in the positive sense. If a function *f* is analytic inside and on *C* except for a finite number of singular points z_k (k= 1,2,….,n) inside *C*, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z) \qquad \text{……………..(i)}$$



Proof:-

To prove the theorem , let the points z_k (k=1,2,…,n) be centers of positively oriented circles ck which are interior to C and are so small that no two of them have points in common. The circles Ck, together with the simple closed contour C form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain consisting of the points inside C and exterior to each C_k. Hence, according to the adaptation of Cauchy-Goursat theorem to such domains.

$$\int_C f(z)dz - \sum_{k=1}^{n} \int_{C_k} f(z)dz=0$$

This residues to equation (i) because

$$\int_{C_k} f(z)dz = 2\pi i\, Res_{z=z_k} f(z) \qquad (k =1,2,….,n)$$

Hence the proof.

EXAMPLE :1

Let us use Cauchy's Residue theorem to evaluate the integral

$$\int_C \frac{5z-2}{z(z-1)}dz$$

where C is the circle |z|=2 described counterclockwise. The integrand has the two isolated singularities z=0 and z=1 both of which are interior to C. we can find the residues B_1 at z=0 and B_2 at z=1 with the aid of the Maclaurin series

$$\frac{1}{1-z} = 1+z+z^2+….. \qquad (|z|<1)$$

we observe first that when 0<|z|<1

$$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z} \cdot \frac{-1}{1-z} = \left(5 - \frac{2}{z}\right)(-1-z-z^2- ……)$$

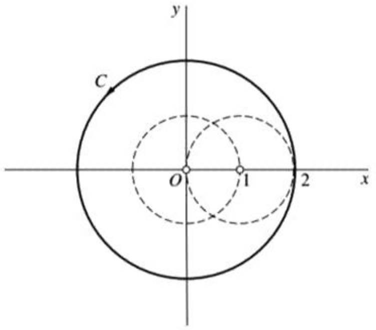
And by indentifying the coefficient of 1/z in the product on the right here, we find that B_1=2 also, since

$$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1} \cdot \frac{1}{1+(z-1)}$$

$$= \left(5+\frac{3}{z-1}\right) [1-(z-1)+(z-1)^2-....]$$

When $0<|z-1|<1$, it is clear that $B_2=3$. Thus

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i\,(B_1+B_2) = 10\pi i$$



in this example it is actually simpler to write the integrand as the sum of its partial functions:

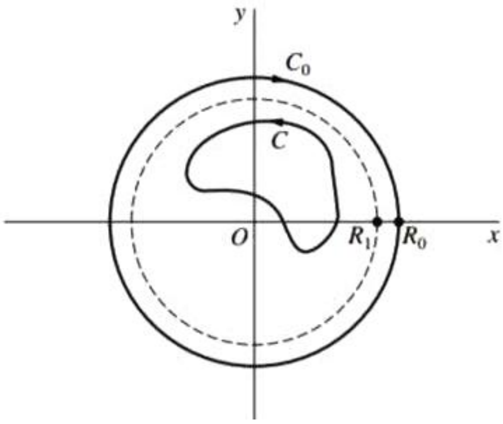
$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{z-1}$$

Then ,since 2/z is already a Lauren series when $0<|z|<1$ and since 3/(z-1) is a Laurent Series when $0<|z-1|<1$. It follows that

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i(2) + 2\pi i(3) = 10\pi i$$

5. Residue at infinity

Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C. next, let R_1 denote a positive number which is large enough that C lies inside the circle $|z|=R_1$. The function f is evidently analytic throughout the domain $R_1<|z|<\infty$, the point at infinity is then said to be an isolated singular point of f.



Now let C_0 denote a circle $|z|=R_0$, oriented in the clockwise direction. Where $R_0 > R_1$. The residue of f at infinity is defined by means of the equation

1. $\int_{C_0} f(z)dz = 2\pi i \operatorname{Res}_{z=\infty} f(z)$

note that the circle C_0 keeps the point at infinity on the left. Just as the singular point in the finite plane is on the left. Since f is analytic throughout the closed region bounded by C and C0, the principle of deformation of paths tell us that

$$\int_C f(z)dz = \int_{-C_0} f(z)dz = -\int_{C_0} f(z)dz$$

so in view of dentition (1)

2. $\int_C f(z)dz = -2\pi i \operatorname{Res}_{z=\infty} f(z)$

to find this residue, writing the Laurent Series

3. $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad (R_1<|z|<\infty)$

where,

4. $C_n = \frac{1}{2\pi i}\int_{-C_0} f(z)dz/z^{n+1} \qquad (n=0,\pm1, \pm2,\ldots\ldots)$

Replacing z by 1/z in expansion (3) and then multiplying through the result by $1/z^2$ we see that

$$(1/z^2)f(\tfrac{1}{z}) = \sum_{n=-\infty}^{\infty} C_n/z^{n+2} = \sum_{n=-\infty}^{\infty} C_{n-2}/z^n \qquad (0<|z|< 1/R_1)$$

and

$$C_{-1} = \operatorname{Res}_{z=0} [(\tfrac{1}{z^2})\, f(\tfrac{1}{z})]$$

Putting n=-1 in expression (4) we have

$$C_{-1} = \frac{1}{2\pi i}\int_{-C_0} f(z)dz$$

Or

5. $\int_{C_0} f(z)dz = -2\pi i \operatorname{Res}_{z=0} [1/z^2 f(1/z)]$

note how it follows from this and definition(1) that

6. $\operatorname{Res}_{z=\infty} f(z) = -\operatorname{Res}_{z=0} [\frac{1}{z^2} f(\frac{1}{z})]$

With equations (2) and (6) , the following theorem is now established. This theorem is sometimes more efficient to use than Cauchy's residue theorem since it involves only one residue.

THEOREM:2 If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C.

Proof:- In the Laurent expansion for f(z) around z_0, for a given k we have

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{k+1}}\, dz$$

so using k = 1, the result follows

the coefficient a_{-1} will be very important for our uses so we give it its own name.

If f(z) = $\sum_{k=-\infty}^{\infty} a_k z^k$
in a deleted neighborhood of z_0, then we call the coefficient a_{-1} the residue of f at z_0 and we denote it by Res(f; z_0).
Evaluation of residues is fairly straight forward and we do not (always) have to find the Laurent expansion explicitly to find residues.

EXAMPLE:1
We evaluate the integral of
f(z) = $\frac{5z-2}{z(z-1)}$
around the cirle |z|=2 described counterclockwise, by finding the residues of f(z) at z=0 and z=1. Since
$\frac{1}{z^2}f(\frac{1}{z}) = \frac{5-2z}{z(1-z)} = \frac{5-2z}{z} \cdot \frac{1}{1-z} = (\frac{5}{z} - 2)\,(1+z+z^2+. . . . +) = \frac{5}{z} +3+3z+. . . .$ (0<|z|<1)
We see that the theorem here can also be used , where the desired residue is 5. More precisely
$\int_C \frac{5z-2}{z(z-1)}\,dz = 2\pi i(5) = 10\pi i$
where C is the circle in question. This is of course the result.

EXAMPLE :2
Consider the function f(z) = 1 + z-1, z≠0. Then
F(w) = f(1/w) =1 +w (w≠0), and lim w→0 F(w) =1.
Thus F(w) has a removable singularity at w=0 and therefore, the point at infinity is a removable singularity of f(z) . Further, Res [f(z);∞] = -1. From this we also observe that if f has a removable singularity at the point at infinity, then the residue of f at ∞ may prove to be non-zero in contrast to the case when f has removable singularity at a finite point.

6. THE THREE TYPES OF ISOLATED SINGULAR POINTS

I. **POLES** :- If z_0 is not a singular point and we can find a positive integer n such that $\lim_{z\to z_0}(z-z_0)^n f(z)$ = A≠0, then z=z_0 is called a Pole of order n. If n=1, z_0 is called a simple pole.

Example:- f(z) = $\frac{1}{(z-2)^3}$ has a pole of order 3 at z=2.

II. **REMOVABLE SINGULARITIES** :- If a single valued function f(z) is not defined at z=a but $\lim_{z\to a} f(z)$ exists, then z=a is called a Removable Singularity. In such case we define f(z) at z=a as equal to $\lim_{z\to a} f(z)$, and f(z) will then be analytic at a.

Example :- If f(z)= $\frac{sinz}{z}$, then z=0 is a removable singularity since f(0) is not defined but $\lim_{z\to a} \frac{sinz}{z}$ = 1.

III. **ESSENTIAL SINGULARITIES** :- If f(z) is a single-valued function, then any singularity which is not a pole or removable singularity is called an essential singularity. If z=a is an essential singularity of f(z) , the principle part of the Laurent Series expansion has infinitely many terms.

Example:- Since $e^{1/z} = 1+ \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$, z=0 is an essential singularity.

7. RESIDUES AT POLES

When a function f has an isolated singularity at a point z_0 ,the basic method for identifying z_0 as a pole and finding the residue there is to write the appropriate Laurent Series and to note the coefficient of $1/(z-z_0)$. The following theorem provides an alternative characterization of poles and a way of finding residues at poles that is often more convenient.

THEOREM :3

An isolated singular point z_0 of a function f is a pole of order m if and only if f(z) can be written in the form

1. $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$

Where ϕ(z) is analytic and nonzero at z_0. Moreover,

2. $Res_{z=z0}\, f(z) = \varphi(z_0)$ if m=1

And

3. $Res_{z=z0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$ if m≥2

Observe that expression need not have been written separately since with the convention that $\varphi(0)(z_0) = \varphi(z_0)$ and 0!=1, expression 3 reduces to it when m=1.

To prove the theorem, we first assume that f(z) has the form (1) and since the $\varphi(z)$ is analytic at z_0, it has a Taylor series representation

$\phi(z) = \phi(z_0) + \frac{\varphi'(z0)}{1!}(z-z_0) + \frac{\varphi''(z_0)}{2!}(z-z_0)^2 + \ldots + \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \sum_{n=m}^{\infty}\frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^n$

In some neighborhood |z-z0|< ε of z0; and from expression (1) it follows that

$f(z) = \frac{\varphi(z_0)}{(z-z_0))^m} + \frac{\varphi'(z_0)/1!}{(z-z_0))^{m-1}} + \frac{\varphi''(z_0)/2!}{(z-z_0))^{m-2}} + \ldots + \frac{\varphi^{(m-1)}(z_0)/(m-1)!}{(z-z_0)} + \sum_{n=m}^{\infty}\frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$

when $0<|z-z_0|< \varepsilon$. This is Laurent series representation, together with the fact that $\phi(z0)\neq 0$ reveals that z_0 is indeed, a pole of order m of f(z). the coefficient of $1/(z-z_0)$ tells us of course that the residue of f(z) at z_0 is as in the statement of the theorem.

Suppose, on the other hand, that we know only that z_0 is a pole of order m of f or that f(z) has a Laurent Series representation

$F(z) = \sum_{n=0}^{\infty} a_n(z-z_0) + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \frac{b_m}{(z-z_0)^m}$ $(b_m\neq 0)$

Which is valid in a punctured disk $0<|z-z_0|<R_2$. The function $\phi(z)$ defined by means of the equations

$\Phi(z) = (z-z_0)^m f(z)$ when $z\neq z_0$

b_m when $z=z_0$

evidently has the power series representation

$\phi(z) = b_m + b_{m-1}(z-z_0) + \ldots + b_2(z-z_0)^{m-2} + b_1(z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n(z-z_0)^{m+n}$

throughout the entire disk $|z-z_0|<R_2$. Consequently, $\phi(z)$ is analytic in that disk and in particular at z_0. Inasmuch as $\phi(z_0) = b_m \neq 0$, expression (1) is established; and the proof of the theorem is complete.

EXAMPLE :1

The function

$f(z) = \frac{z+1}{z^2+9}$

Has an isolated singular point at z=3i and can be written

$f(z) = \frac{\varphi(z)}{z-3i}$ where $\phi(z) = \frac{z+1}{z+3i}$

Since $\phi(z)$ is analytic at z=3i and $\phi(3i)\neq 0$, that point is a simple pole of the function f; and the residue there is

$B_1 = \phi(3i)= \frac{3i+1}{6i} \cdot \frac{-i}{-i} = \frac{3-i}{6}$

The point z=-3i is also a simple pole of f, with residue

$B_2 = \frac{3+i}{6}$

EXAMPLE:2

If $f(z)= \frac{z^3+2z}{(z-i)^3}$

Then $f(z)= \frac{\varphi(z)}{(z-i)^3}$ where $\phi(z) = z^3+2z$

The function ϕ(z) is entire and ϕ(i)=i≠0. Hence f has a pole of order 3 at z=i, with residue

$$B= \frac{\varphi''(i)}{2!} = \frac{6i}{2!} = 3i$$

The theorem can of course, be used when branches of multiple-valued functions are involved.

EXAMPLE: 3

If for instance, the residue of the function

$$F(z) = \frac{\sinh z}{z^4}$$

Is needed at the singularity z=0, it would be incorrect to write

$F(z) = \frac{\varphi(z)}{z^4}$ where ϕ(z)=sinh z

And to attempt an application of formula with m=4. For it is necessary that $\phi(z_0) \neq 0$ if that formula is to be used. In this case, the simplest way to find the residue is to write out a few terms of the Laurent series for f(z). there it is shown that z=0 is a pole of the third order, with residue B=1/6.

8. FURTHER STUDIES

Next we can study the behavior of function near poles.

9. BIBILOGRAPHY

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